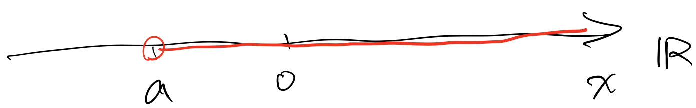
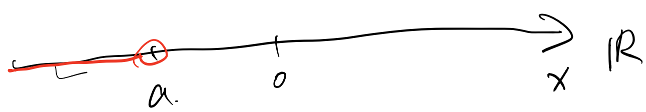
E.g. The set consisting of three elements: 1, 2, 3 is written as $\{1, 2, 3\}$

E.g. $\{0, 2\} = \{x \mid x(x-2) = 0\}$

E.g. $(a, \infty) = \{x \mid x > a\}$



E.g. $(-\infty, a] = \{x \mid, x \leq a\}$

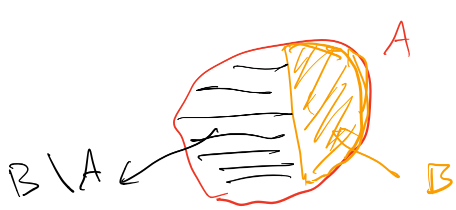


Definition $A \subset B$ A is a subset of B

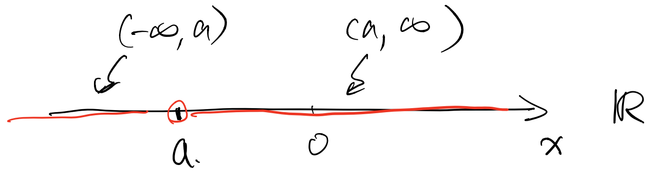
$B \setminus A = \{x \mid x \in B, x \notin A\}$

↑

the "complement" of A in B

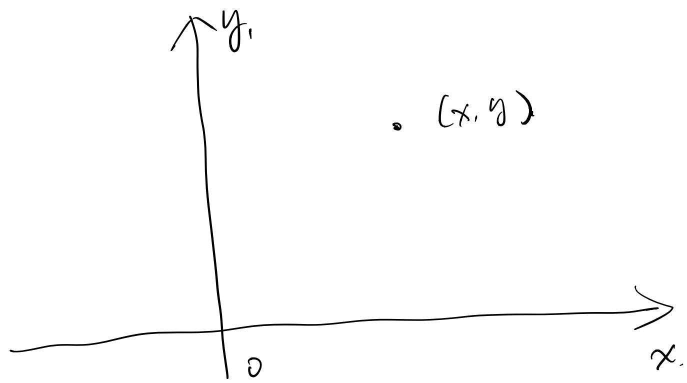


E.g. $\mathbb{R} \setminus \{a\} = (-\infty, a) \cup (a, \infty)$



Def. A function $f: A \to B$ is also called a "map" from A to B.
A is the "source" of the map f
B is the "target" of the map f

§ Visualizing a real-valued function of one-variable using its graph in the x-y plane

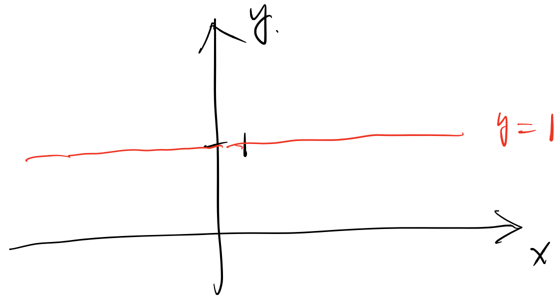


The "graph" of the real-valued function of one variable $f(x)$

$$\Gamma(f) = \{ (x, y) \mid y = f(x) \}$$

E.g. A "constant function"

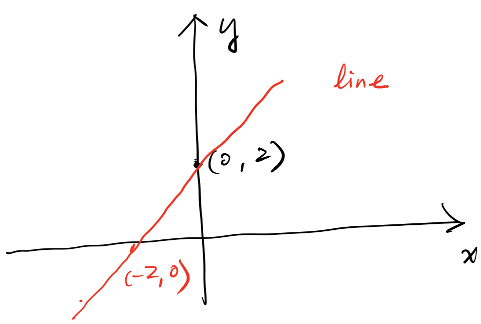
$f(x) = 1$



E.g. A "linear function" is a function of the form

$f(x) = ax + b$ where $a, b \in \mathbb{R}$

e.g. $f(x) = x + 2$



MATH1520C University Mathematics for Applications	Spring 2021

Chapter 1: Notation and Functions

Learning Objectives:
(1) Identify the domain of a function, and evaluate a function from an equation.
(2) Gain familiarity with piecewise functions.
(3) Study the vertical line test.
(4) Know how to form and use composite functions.

1.1 Set

- **Set** is a collection of objects (called **elements**)
  1. Order of elements does not matter. E.g. $\{1, 2, 3\} = \{3, 2, 1\}$.
  2. Representation of a set is not unique. E.g. $\{-2, 2\} = \{x \mid x^2 = 4\}$.
- $\in$: belongs to. If a is an element of A, we say that a belongs to A; denoted as $a \in A$.
- $\subset$: subset of. Let A, B be two sets such that $\forall a \in A$, $a \in B$. Then we say that A is a subset of B; denoted as $A \subset B$.

  for all

Remark. $A \subset B$ is sometimes written as $A \subseteq B$ to emphasize the fact that $A = B$ is a possibility. If $A \subset B$ but $A \neq B$, then A is said to be a *proper subset* (or a strict subset) of B, written as $A \Subset B$.

$A \subset B \Leftrightarrow B \supset A$: B is a *supset* of A.

Example 1.1.1.

1. $A = \{1, 2, 3\}$, $B = \{2, 3, 5, 7\}$, $C = \{1, 2, 3, 4, 5\}$. Then $A \subseteq C$ (in fact $A \Subset C$), $1 \in A$, but $1 \notin B$ and $B \not\subseteq C$. $1 \in C$, but $1 \notin B$
2. $C =$ the set of all students studying at CUHK. $M =$ the set of all math major students currently studying at CUHK. Then $M \subseteq C$. You $\in C$.

Example 1.1.2. Some important number sets:

1. $\mathbb{N}$: the set of all natural numbers (positive integers) $= \{1, 2, 3, \ldots\}$.
2. $\mathbb{Z}$: the set of all integers $= \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$. $= \{0, \pm 1, \pm 2, \cdots\}$
3. $\mathbb{Q}$: the set of all rational numbers $= \{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\}$.

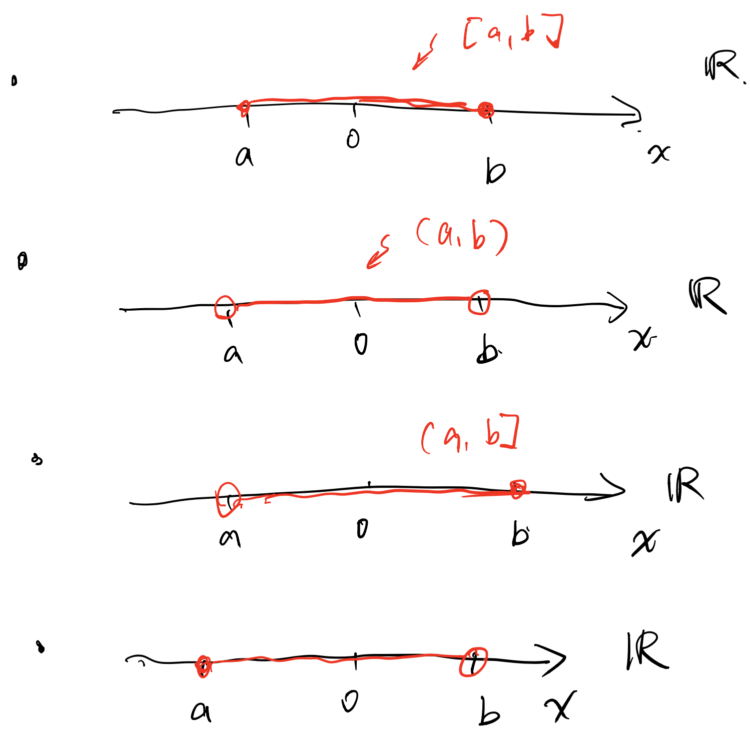
4. $\mathbb{R}$: the set of all real numbers.

Remark. If the elements in a set can be ordered and the ordering are taken into account in the definition, then it is called an *ordered set*. E.g. $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ may call be viewed as ordered sets.

1.2 Intervals

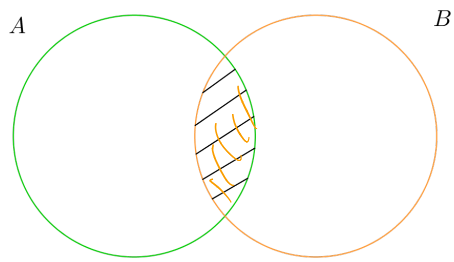
- $[a, b] = \{x \,|\, a \le x \le b\}$. (closed interval)
- $(a, b) = \{x \,|\, a < x < b\}$. (open interval)
- $(a, b] = \{x \,|\, a < x \le b\}$.
- $[a, \infty)$: the set of all real numbers x such that $a \le x$.

Drawing open/closed intervals on the real line:

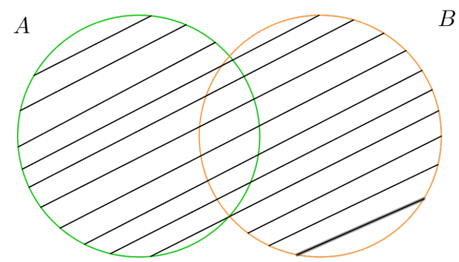


1.3 Set operations

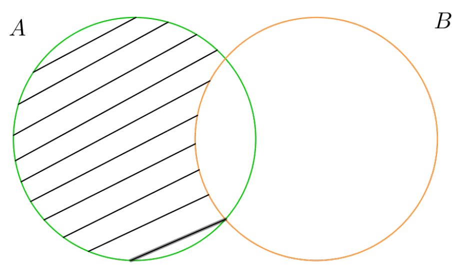
Let A, B be two sets:



Intersection
$A \cap B = \{x \,|\, x \in A \;\; \text{and} \;\; x \in B\}$

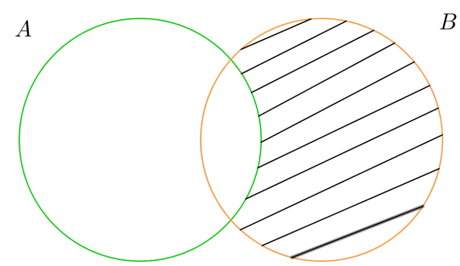


Union
$A \cup B = \{x \,|\, x \in A \;\; \text{or} \;\; x \in B\}$



Relative complement of B in A
$A \backslash B = \{x \,|\, x \in A \;\; \text{and} \;\; x \notin B\}$

Alternative notation: $A - B$



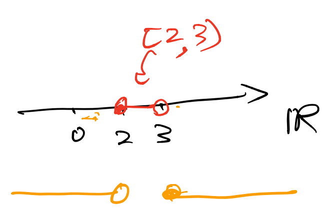
Relative complement of A in B
$B \backslash A = \{x \,|\, x \in B \;\; \text{and} \;\; x \notin A\}$ $= B - A$

Example 1.3.1.

1. Let $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$, $C = \{5\}$.
   $A \cap B = \{2, 3\}$, $A \cup B = \{1, 2, 3, 4\}$, $A \backslash B = \{1\}$, $B \backslash A = \{4\}$, $A \backslash C = A$.
   $A \backslash (A \cap B)$
2. $\mathbb{R} \backslash \{a\}$: the set of all real numbers x, except $x = a$.

Exercise 1.3.1.

1. What are the meanings of the following sets
   (a) $(-\infty, a)$.
   (b) $\mathbb{R} \backslash \{1, 2, 3\}$
   (c) $\mathbb{R} \backslash [2, 3)$. $= (-\infty, 2) \cup [3, \infty)$
2. Show that $\mathbb{R} \backslash [1, \infty) = (-\infty, 1)$.



$\mathbb{R} \backslash [2, 3)$

1.4 Functions

Definition 1.4.1. A **function** is a rule that assigns to EACH element x in a set A EXACTLY ONE element y in a set B. If the function is denoted by f, then we may write

$$f : A \to B.$$

also called 'f is a "map" from A to B'

The set A is called the domain of the function. The set B is called the codomain of f. The assigned elements in B is called the range of f.

$f(x) = y$

x is the independent variable of f, and y is the dependent variable of f.

Given $a \in A$, $f(a) \in B$ is said to be the *value* of the function f at a. Given $S \subset A$,

subset of A.

$$f(S) := \{f(a) \mid a \in S\} \subset B$$

is said to be the *image* of S under f. In particular, the "range" of f, as defined above, is $f(A) \subset B$.

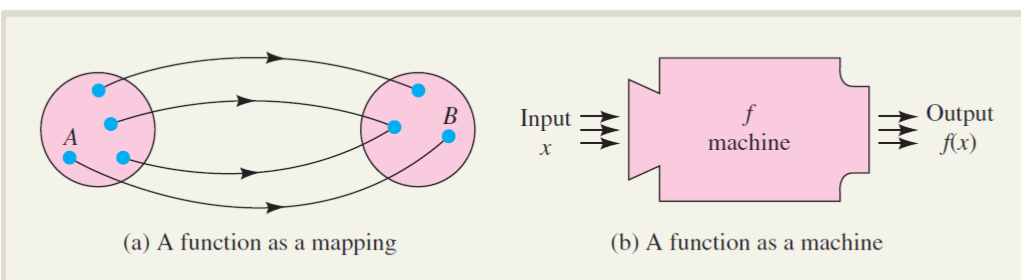
When the domain and range of a function are both sets of real numbers, the function is said to be a real-valued function of one variable, and we write

$$f : \mathbb{R} \to \mathbb{R}.$$

x

Most functions encountered in this course are real-valued functions of one variable. Unless otherwise specified, a function is a real-valued function of one variable in this course.

Remark. There is some ambiguity in the definition of "range" in math literature. See the Wiki article.



(a) A function as a mapping (b) A function as a machine

Example 1.4.1. $f : [-1, 3) \to \mathbb{R}$ is defined by $f(x) = x^2 + 4$ (sometimes written as $y = x^2 + 4$). Then

$$f(0) = (0)^2 + 4 = 4.$$

domain $= [-1, 3)$, codomain $= \mathbb{R}$, range of f $= [4, 13)$.

$3^2 + 4$

when x = 0

Remark. If a function is given by a formula without domain specified, then assume domain = set of all x for which $f(x)$ is well defined, this domain is also called the natural domain of f.

Example 1.4.2. Find the natural domain of the functions.

1. $f(x) = \dfrac{1}{x-3}$. this formula makes sense $\forall x \in \mathbb{R}$ except for $x=3$. : Natural domain for f. is $\mathbb{R} \setminus \{3\}$
2. $g(t) = \dfrac{\sqrt{3-2t}}{t^2+4}$.

Solution. $t \in \mathbb{R}$, the denominator t^2+4 can never be 0
$\sqrt{3-2t}$ is not well-defined when $3-2t < 0$

1. $\dfrac{1}{x-3}$ is not defined when its denominator $x-3=0$, i.e. $x=3$. So the domain is $\mathbb{R}\setminus\{3\}$.
2. The domain of $\sqrt{3-2t}$ consists of all x such that $3-2t \geq 0$, which implies that $t \leq \frac{3}{2}$. Hence the domain is $(-\infty, \frac{3}{2}]$.

■

$$x^2-1 = (x-1)(x+1)$$

Example 1.4.3. Let $f(x) = \dfrac{x^2-1}{x-1}$ and $g(x) = x+1$. Can we say f and g are the same function?

$$= \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}} = x+1$$

Solution. No! The domain of $f(x)$ is $\mathbb{R}\setminus\{1\}$, the domain of $g(x)$ is $\mathbb{R}$.
Only when $x \neq 1$, $f(x) = g(x)$. ■

is well-defined only when $x-1 \neq 0$

f

1.4.1 Vertical Line Test for Graph

A way to visualize a function is its graph. If f is a real-valued function of one variable, its graph consists of the points in the Cartesian plane $\mathbb{R}^2$ whose coordinates are the input-output pairs for f. In set notation, the graph is

$$\Gamma(f) = \{(x, y) \in \mathbb{R}^2 : x \in \mathbb{R}, y = f(x)\}.$$

Review: Graphing a real-valued function of one variable: [HBSP] 1.2.

Example 1.4.4. linear functions; piecewise linear functions; quadractic functions, exponential and log functions, trig functions.